

SUBJECT: MATHEMATICS

**PAPER: ADVANCED MATHEMATICAL
METHODS**

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Fourier series in complex form

Fourier series of a function $f(x)$ of period $2l$ is

$$f(x) = \frac{1}{2}a_0 + a_1 \cos \frac{\pi x}{l} + a_2 \cos \frac{2\pi x}{l} + \dots + a_n \cos \frac{n\pi x}{l} + \dots \\ + b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + \dots + b_n \sin \frac{n\pi x}{l} + \dots \quad (1)$$

We know that

$$\cos x = \frac{1}{2}[e^{ix} + e^{-ix}], \quad \sin x = \frac{1}{2i}[e^{ix} - e^{-ix}]$$

On putting the values of $\cos x$ and $\sin x$ in (1), we get

$$f(x) = \frac{1}{2}a_0 + \frac{a_1}{2}\left[e^{i\frac{\pi x}{l}} + e^{-i\frac{\pi x}{l}}\right] + \frac{a_2}{2}\left[e^{i\frac{2\pi x}{l}} + e^{-i\frac{2\pi x}{l}}\right] + \dots \\ + \frac{a_n}{2}\left[e^{i\frac{n\pi x}{l}} + e^{-i\frac{n\pi x}{l}}\right] + \frac{b_1}{2i}\left[e^{i\frac{\pi x}{l}} - e^{-i\frac{\pi x}{l}}\right] + \frac{b_2}{2i}\left[e^{i\frac{2\pi x}{l}} - e^{-i\frac{2\pi x}{l}}\right] \\ + \dots + \frac{b_n}{2i}\left[e^{i\frac{n\pi x}{l}} - e^{-i\frac{n\pi x}{l}}\right] + \dots \\ f(x) = \frac{1}{2}a_0 + \left(\frac{a_1}{2} - i\frac{b_1}{2}\right)e^{i\frac{\pi x}{l}} + \left(\frac{a_2}{2} - i\frac{b_2}{2}\right)e^{i\frac{2\pi x}{l}} + \dots + \left(\frac{a_n}{2} - i\frac{b_n}{2}\right)e^{i\frac{n\pi x}{l}} \\ + \left(\frac{a_1}{2} + i\frac{b_1}{2}\right)e^{-i\frac{\pi x}{l}} + \left(\frac{a_2}{2} + i\frac{b_2}{2}\right)e^{-i\frac{2\pi x}{l}} + \dots + \left(\frac{a_n}{2} + i\frac{b_n}{2}\right)e^{-i\frac{n\pi x}{l}} \\ = c_0 + c_1 e^{i\frac{\pi x}{l}} + c_2 e^{i\frac{2\pi x}{l}} + \dots + c_n e^{i\frac{n\pi x}{l}} + c_{-1} e^{-i\frac{\pi x}{l}} + c_{-2} e^{-i\frac{2\pi x}{l}} \\ + \dots + c_{-n} e^{-i\frac{n\pi x}{l}} + \dots$$

$$f(x) = c_0 + \sum c_n e^{i\frac{n\pi x}{l}} + \sum c_{-n} e^{-i\frac{n\pi x}{l}}$$

$$\text{where } c_0 = \frac{1}{2}a_0, \quad c_n = \frac{1}{2}(a_n - ib_n), \quad c_{-n} = \frac{1}{2}(a_n + ib_n)$$

$$c_0 = \frac{1}{2}a_0 = \frac{1}{2} \cdot \frac{1}{l} \int_0^{2l} f(x) dx, \quad a_n = \frac{1}{l} \int_0^{2l} f(x) \cos \frac{n\pi x}{l} dx$$

$$b_n = \frac{1}{l} \int_0^{2l} f(x) \sin \frac{n\pi x}{l} dx$$

$$i(a_n - ib_n) = \left(\frac{1}{l} \int_0^{2l} f(x) \cos \frac{n\pi x}{l} dx \right) - i \left(\frac{1}{l} \int_0^{2l} f(x) \sin \frac{n\pi x}{l} dx \right)$$

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$$\frac{a_n - i b_n}{2} = \frac{1}{2\ell} \int_0^{2\ell} f(x) \cdot \left(\frac{\cos \frac{n\pi x}{\ell} - i \sin \frac{n\pi x}{\ell}}{2} \right) dx$$

$$\Rightarrow c_n = \frac{1}{2\ell} \int_0^{2\ell} f(x) \cdot e^{-i \frac{n\pi x}{\ell}} dx$$

$$\underline{c}_n = \frac{1}{2\ell} \int_0^{2\ell} f(x) e^{i \frac{n\pi x}{\ell}} dx$$

Example

1. obtain the complex form of fourier series

$$f(x) = \begin{cases} 0, & -\pi \leq x \leq 0 \\ 1, & 0 \leq x \leq \pi \end{cases}$$

Soln. Given $f(x) = \begin{cases} 0, & -\pi \leq x \leq 0 \\ 1, & 0 \leq x \leq \pi \end{cases}$

$$c_0 = \frac{1}{2} \cdot \frac{1}{\ell} \int_0^{2\ell} f(x) dx = \frac{1}{2} \cdot \frac{1}{\pi} \int_0^{2\pi} f(x) dx$$

$$= \frac{1}{2} \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{2\pi} \int_{-\pi}^0 f(x) dx + \frac{1}{2\pi} \int_0^{\pi} f(x) dx$$

$$c_0 = \frac{1}{2\pi} \left[\int_{-\pi}^0 0 \cdot dx + \int_0^{\pi} 1 \cdot dx \right] = \frac{1}{2}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-in x} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^0 f(x) e^{-in x} dx + \frac{1}{2\pi} \int_0^{\pi} f(x) e^{-in x} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^0 0 \cdot e^{-in x} dx + \frac{1}{2\pi} \int_0^{\pi} e^{-in x} dx$$

$$= \frac{1}{2\pi} \left(\frac{e^{-in x}}{-in} \right)_0^{\pi} = \frac{1}{2\pi} \cdot \frac{1}{-in} \left[e^{-in\pi} - 1 \right]$$

$$= \frac{1}{2\pi n} (i) [\cos n\pi - i \sin n\pi - 1] = i \left[\frac{\cos n\pi - 1}{2n\pi} \right]$$

$$c_n = -\frac{i}{2n\pi} \text{ if } n \text{ is odd} \quad \& \quad c_n = 0 \text{ if } n \text{ is even}$$

if $f(x) = \frac{1}{2} a_0 + c_n e^{-in x} + \underline{c}_n e^{in x}$, $a_0, c_n, \underline{c}_n$ is just calculated above.

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Problems for Practice

Find the complex form of

① $f(x) = e^{-x}, -1 \leq x \leq 1$ (2)

Ans $f(x) = \sum \frac{(-1)^n (1 - i n \pi) \sinh 1 \cdot e^{i n \pi x}}{1 + n^2 \pi^2}$

② $f(x) = e^{ax}, -1 < x < 1$

Ans $f(x) = \frac{2}{\pi} - \frac{2}{\pi} \left[\frac{e^{2a} + e^{-2a}}{1.3} + \frac{e^{4a} + e^{-4a}}{3.5} + \dots \right]$

③ $f(x) = \cos ax, -\pi < x < \pi$

Ans $f(x) = \frac{a}{\pi} \sin a\pi + \sum_{n=-\infty}^{\infty} \frac{(-1)^n e^{ian}}{a^2 - n^2}$

Practical Harmonic analysis

Some time $f(x)$ is not given only arguments and entries have been given. In this case the process of finding the Fourier series is called Harmonic analysis. Here a_0, a_n, b_n of $f(x) = \frac{a_0}{2} + \sum (a_n \cos nx + b_n \sin nx)$ can be calculated as

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx = 2 \cdot \frac{1}{2\pi - 0} \int_0^{2\pi} f(x) dx$$

$$[\therefore \text{mean} = \frac{1}{b-a} \int_a^b f(x) dx]$$

$$\Rightarrow \boxed{a_0 = 2 [\text{mean value of } f(x) \text{ in }] 0, 2\pi []}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx dx = 2 \cdot \frac{1}{2\pi - 0} \int_0^{2\pi} f(x) \cos nx dx$$

$$= 2 \cdot [\text{mean value of } f(x) \cos nx \text{ in }] 0, 2\pi (]$$

Similarly

$$b_n = 2 [\text{mean value of } f(x) \sin nx \text{ in }] 0, 2\pi (]$$

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Remember In Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\text{First harmonic} = a_1 \cos x + b_1 \sin x$$

$$\text{Second harmonic} = a_2 \cos 2x + b_2 \sin 2x$$

$$\vdots$$

$$n^{\text{th}} \text{ harmonic} = a_n \cos nx + b_n \sin nx$$

Example. Find Fourier series as far as the second harmonic to represent the function given by table below

x	0	30	60	90	120	150	180	210	240	270	300	330
$f(x)$	2.34	3.01	3.69	4.15	3.69	2.20	0.83	0.51	0.88	1.09	1.19	1.64

Solu.

x°	$\sin x$	$\sin 2x$	$\cos x$	$\cos 2x$	$f(x)$	$f(x) \cdot \sin x$	$f(x) \cdot \sin 2x$	$f(x) \cdot \cos x$	$f(x) \cdot \cos 2x$
0	0	0	1	1	2.34	0	0	2.340	2.340
30	0.50	0.87	0.87	0.50	3.01	1.505	2.619	2.619	1.505
60	0.87	0.87	0.50	-0.50	3.69	3.210	3.210	1.845	-1.845
90	1.00	0	0	-1.00	4.15	4.150	0	0	-4.150
120	0.87	-0.87	-0.50	-0.50	3.69	3.210	-3.210	-1.845	-1.845
150	0.50	-0.87	-0.87	0.50	2.20	1.100	-1.914	-1.914	1.100
180	0	0	-1	1.00	0.83	0	0	-0.830	0.830
210	-0.50	0.87	-0.87	0.50	0.51	-0.255	0.444	-0.444	0.255
240	-0.87	0.87	-0.50	-0.50	0.88	-0.766	0.766	-0.440	-0.440
270	-1.00	0	0	-1.00	1.09	-1.090	0	0	-1.090
300	-0.87	-0.87	0.50	-0.50	1.19	-1.035	-1.035	0.595	-0.595
330	-0.50	-0.87	0.87	0.50	1.64	-0.820	-1.427	1.427	0.820
					25.22	9.209	-0.547	3.253	-3.115

$$a_0 = 2 \times \text{mean of } f(x) = 2 \times \frac{25.22}{12} = 4.203$$

$$a_1 = 2 \times \text{mean of } f(x) \cos x = 2 \times \frac{3.353}{12} = 0.559$$

$$a_2 = 2 \times \text{mean of } f(x) \cos 2x = 2 \times \frac{-3.115}{12} = -0.519$$

$$b_1 = 2 \times \text{mean of } f(x) \sin x = 2 \times \frac{9.209}{12} = 1.535$$

$$b_2 = 2 \times \text{mean of } f(x) \sin 2x = 2 \times \frac{-0.547}{12} = -0.091$$

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$$b_2 = 2 \times \text{mean of } f(x) \sin 2x \\ = 2 \times \frac{-0.547}{12} = -0.091$$

Fourier series is

$$f(x) = \frac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + \dots + b_1 \sin x + b_2 \sin 2x + \dots \\ = 2.1015 + 0.559 \cos x - 0.519 \cos 2x + \dots \\ + 1.535 \sin x - 0.091 \sin 2x + \dots$$

Dirichlet's conditions for a Fourier series.

of the function $f(x)$ for the interval $(-\pi, \pi)$

(1) is single valued (2) is bounded

(3) has at most a finite number of maxima and minima

(4) has only a finite no of discontinuities.

(5) is $f(x+2\pi) = f(x)$ for values of x outside $[-\pi, \pi]$ then

$$S_p(x) = \frac{1}{2}a_0 + \sum_{n=1}^p a_n \cos nx + \sum_{n=1}^p b_n \sin nx$$

converges to $f(x)$ as $p \rightarrow \infty$ at values of x for which $f(x)$ is continuous and to

$$\frac{1}{2} [f(x+0) + f(x-0)]$$

i.e. Suppose $f(x)$ is discontinuous at $x=c$ then

$$f(c) = \frac{1}{2} \left[\lim_{h \rightarrow 0} f(c+h) + \lim_{h \rightarrow 0} f(c-h) \right]$$

$$\text{where } f(c) = \lim_{x \rightarrow c} f(x) = \left[\frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx \right]_{x=c}$$

Advantages of Fourier Series

- ① Discontinuous function can be represented by Fourier series. Although derivatives of the discontinuous functions do not exist while it is not true for Taylor's series case.
- ② The Fourier series is useful in expanding the periodic functions since outside the closed interval there exists a periodic extension of the function.
- ③ Expansion of an oscillation function by Fourier series gives all modes of oscillation, fundamental and all overtones which is extremely useful in physics.
- ④ Fourier series of discontinuous function is not uniformly convergent at all points.
- ⑤ Term by term integration of a convergent Fourier series is always valid and it could be valid if the series is not convergent. However term by term, differentiation of a Fourier series is not valid in most cases.

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